INDIAN STATISTICAL INSTITUTE
Mid-Semestral Exam
Introduction to Representation Theory
2018-2019

Total marks: 60
Time: 3 hours

In the following questions you may assume all representations to be over $\mathbb{C}$, and G to be a finite group .

1. (a) If V^* denotes the dual vector space of V, prove that $V^* \otimes W \cong \mathrm{Hom}_{\mathbb{C}}(V, W)$, as complex vector spaces.
   (b) Let $\rho_1 : G \longrightarrow GL(V)$ and $\rho_2 : G \longrightarrow GL(W)$ be representations of G. Prove that both $V^* \otimes W$ and $\mathrm{Hom}_{\mathbb{C}}(V, W)$ are G-representations, and the above isomorphism is an isomorphism of G-representations. (4+8)

2. (a) State and prove Schur's lemma.
   (b) Deduce that if G is an abelian group, then any irreducible representation of G has degree one. (8+4)

3. (a) Show that the representation $\rho : S_3 \longrightarrow GL_2(\mathbb{C})$ given by
   $$\rho_{(12)} = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}$$
   and
   $$\rho_{(123)} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$$
   is an irreducible representation.
   (b) Write down all the finite dimensional inequivalent irreducible representations of S_3. Justify why these are irreducible and also why this forms the complete list of such representations.
   (c) Draw the character table for S_3. (2+5+5)

4. (a) Define regular representation of a group G.
   (b) Let L denote the regular representation of a group G, $\{\phi^{(1)}, \ldots, \phi^{(s)}\}$ be the complete list of inequivalent irreducible representations of G, and $d_i = \deg \phi^{(i)}$.
   (i) Find the character of L.
   (ii) Also show that the following decomposition holds:
   $$L \sim d_1\phi^{(1)} \oplus d_2\phi^{(2)} \oplus \cdots \oplus d_s\phi^{(s)}.$$
   (2+5+5)

5. Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ be the Klein's 4-group, where $\mathbb{Z}_2$ is the cyclic group of order 2.
   (a) Describe all the irreducible representations of G over $\mathbb{C}$.
   (b) Draw the character table of G. (6+6)